★ Video 3 Chapter 1 Section 1 Other Geometries!

Other Geometries

Timeline – 19th century

Context! mathematics explosion in Europe

NonEuclidean Geometries — ones we'll look at ... there are lots more

3 Point

Flexible

7 Point, Fano's

Klein Disc

Spherical

Hyperbolic

The Three-point Geometry

Undefined Terms: point, line, on

Axioms:

A1 There are exactly 3 distinct points

A2 Two distinct points are on exactly one line.

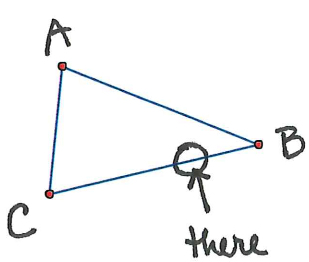
A3 Not all the points are on the same line.

A4. Each pair of distinct lines are on at least one point.

Definitions: Lines that share a point are intersecting.
Lines that do not share a point are parallel.

Model:

Label the points A, B, and C
Check the axioms and definition.



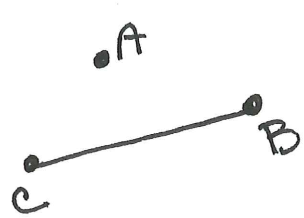
What is exactly 1/3 of the way between B and C?

Impossible? Draw me a unicorn.

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

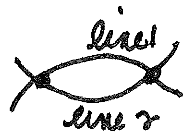
eg



really?

Theorem 1: Each pair of distinct lines is on exactly one point.

Proof of Theorem 1



Suppose there's a pair of lines on more than one point. This cannot be because then the two lines have at least two distinct points on each of them and Axiom 1 states that "two distinct points are on exactly one line".

Thus our supposition cannot be and the theorem is true. QED

Part A
→ suppose 2
→ suppose 4
part B

Theorem 2: There are exactly 3 distinct lines in this geometry.

Popper 1.1, Question 6

Some geometries have lines that are not made up of an infinite number of points.

A. True

B. False

One of the tinier ones. A Flexible Geometry is smaller…just 2 axioms! More more complicated models. Let's think about the Three Point just a little longer…

Could this geometry be called non-Euclidean? Let's come up with some reasons why?

only 3 points
only 3 lines
only 4 axioms
no way to measure anything!

lines have only 2 points
no parallel lines for our setup * none!

Why is this a geometry?
structure! undefined terms (name doesn't count!)

Let's do these two together

Popper 1.1, Question Seven

Why is this a geometry? It has the right structure and the right kind of undefined terms.

A.* True

Popper 1.1, Question Eight

There are axiomatic structures for other math systems like probability and logic. They have the same structure and different undefined terms.

A.* True

The Seven-point geometry

Also known as Fano's geometry. (Gino Fano, published 1892, 1871 – 1952, Italian)

Undefined terms: point, line, on

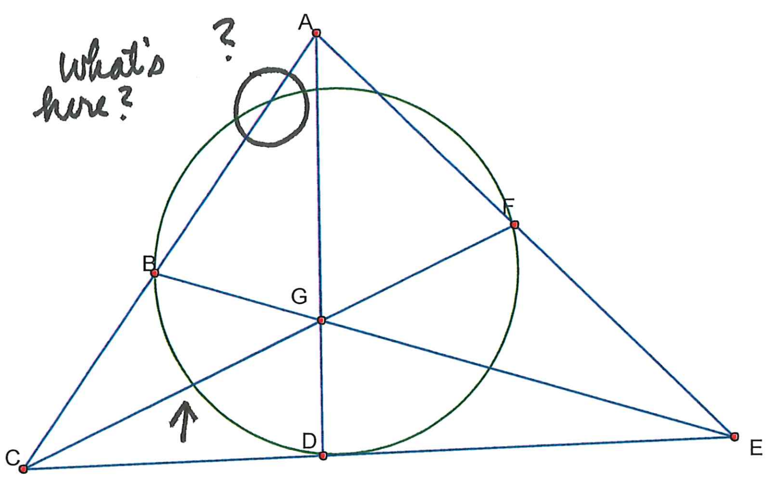
A1: There exists at least one line.

A2: There exist exactly three points on each line.

A3: Not all the points are on the same line.

A4: There exists exactly one line on any two distinct points.

A5: There exists at least one point on any two distinct lines.



Model: How many points and lines?

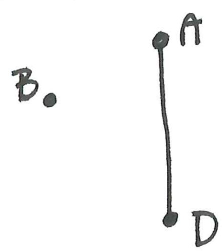
{BDF} is a line! Nobody said "straight" in the axioms!
Where does {BDF} intersect {CBA}? Not to mention:

3 lines on @ point
EG • how many lines on a point? concurrent!

How many points on each line? 3! pretty nonE!

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

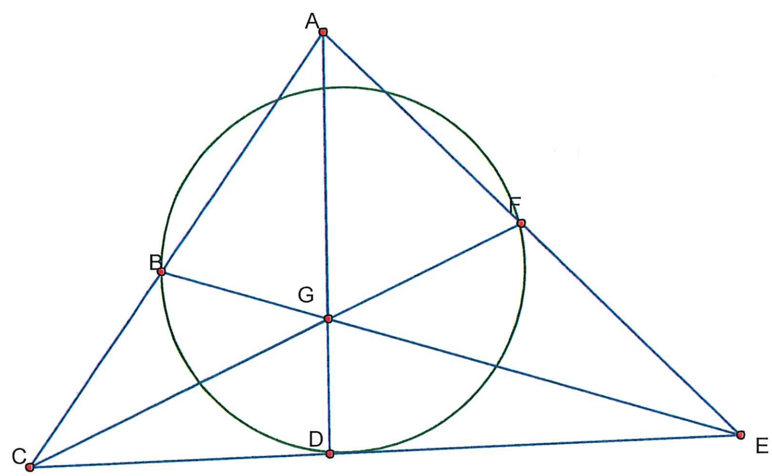


lines thru B

CBA FBD BGE

7 points and 7 lines…what's the situation with respect to parallel lines?

none!



Alternate axioms for Fano's Geometry:

Undefined terms: point, line, on

Axioms:

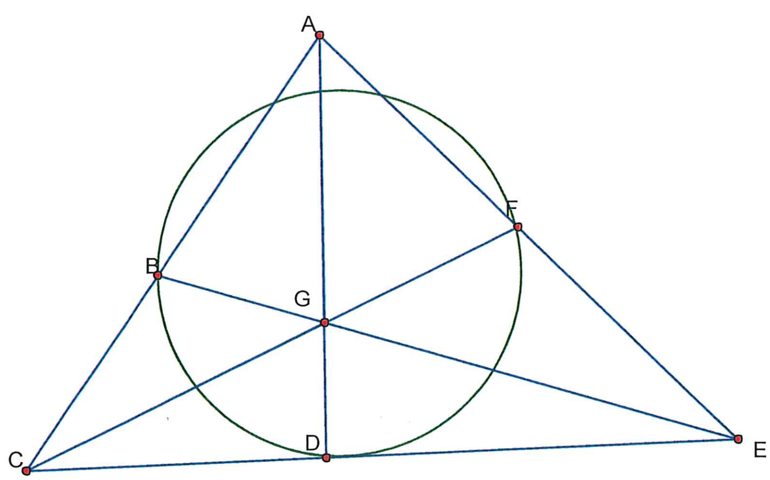
A1 For every pair of distinct points P and Q, there exists exactly one line *l* such that both P and Q lie on that line.

Note that the axiom uses all 3 undefined terms and is defining a relationship among them.

A2 For every line *l* there exist at least 2 distinct points P and Q such that both P and Q lie on the line *l*.

A3 There exist three points that do not all lie on any one line.

Note the "at least 2" in A2!



categorical

EG is

HG is

SG is

Sometimes MORE THAN ONE list of axioms generates the SAME Geometry. Euclidean is categorical though.

Theorems in the alternate axiom system:

1 Two distinct lines have exactly one point in common.

2 There are exactly 7 points in this geometry.

Popper 1.1, Question 9

Why is Fano's Geometry non-Euclidean?

I. Lines are not made up of an infinite number of points.

II. There are only 7 points total.

III. There are only 7 lines total.

IV. All of the above.

V. None of the above.

A. II and III only

B. I only

C. IV

Enough with finite geometries – there's an infinite number of them!

In fact, let's talk about how many there are:

Is there a geometry with 17 points? 1927 points (why did I pick that number?)

N points?

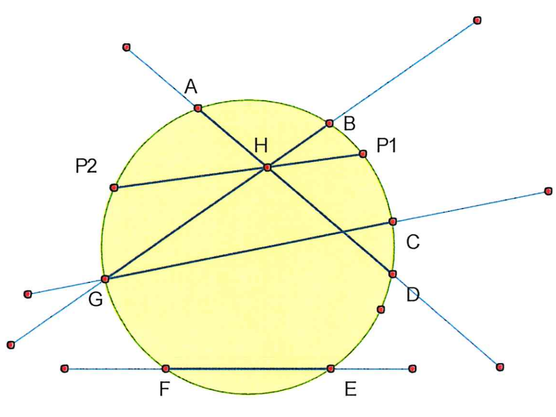
Let's move on to other geometries with an infinite number of points.

The Klein Disk (Felix Klein, 1849 – 1925, German)

Points will be $\{(x, y) \mid x^2 + y^2 < 1\}$, the interior of the Unit Circle, and lines will be the set of all lines that intersect the **interior** of this circle. Point, line, and on has the usual Euclidean sense. Incidence axioms are below

So our model is a proper subset of the Euclidean Plane.

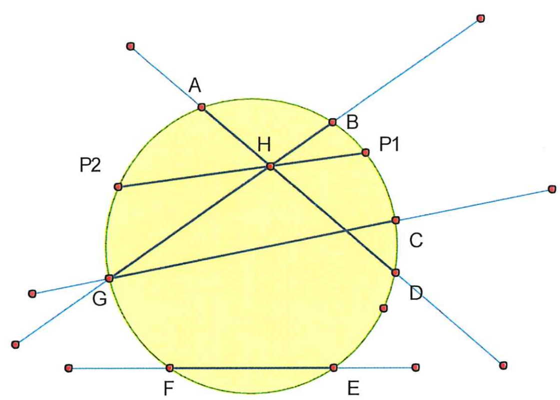
Model:



Note that the labeled points (except H) are NOT points in the geometry. A is on the circle not an interior point. It is convenient to use it, though.

H is a point in the circle's interior and IS a point in the geometry.

We cannot list the number of lines – there are an infinite number of them. A, B, P1, P2, G, C, D, F, and E are not in our space. But the open segments they create are in our space.



Is everybody clear on what is and is not in our space?

Popper 1.1, Question 10

In the Klein Disc, lines are made up of an infinite number of points because they are subsets of traditional Euclidean lines.

A. True

B. False

Checking the axioms:

Undefined terms: point, line, on

Axioms:

IA1 For every pair of distinct points P and Q, there exists exactly one line *l* such that both P and Q lie on that line.

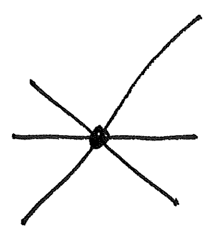
Inheriting…

IA2 For every line *l* there exist at least 2 distinct points P and Q such that both P and Q lie on the line *l*.

Inheriting…

IA3 There exist three points that do not all lie on any one line.

Inheriting…



concurrent lines

Definitions:

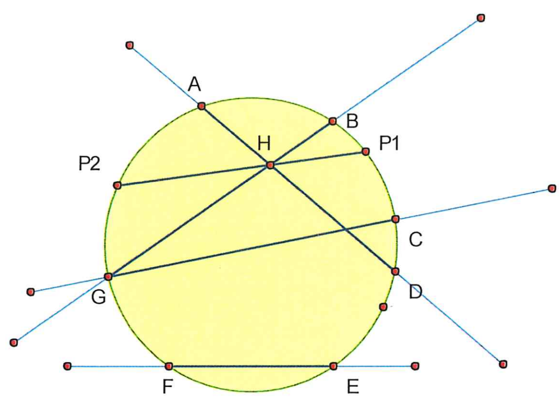
Collinear: Three points, A, B, and C, are said to be collinear if there exists one line l such that all three of the points lie on that line.

Parallel: Lines that share no points are said to be parallel.

SMSG A16 The Parallel Postulate:

Through a given external point there is at most one line parallel to a given line.

We will pick line GC and point H. Now look at line GB.



Let's look at lines GC and GB. They intersect at G…which is NOT a point in the geometry. So GC and GB are parallel. In fact, they are what is called **asymptotically parallel**. They really do share no points in our space.

Now look at P1P2. It, too, is parallel to GC. Furthermore both P1P2 and GB pass through point H. P1P2 is **divergently parallel** to GC.

Not only is the situation vis a vis parallel lines different, we even have flavors of parallel:

asymptotic and divergent. So we are truly non-euclidean here, folks.

Theorem 1: If two distinct lines intersect, then the intersection is exactly one point.

AD & GC eg

Inherited from Euclidean Geometry.

Theorem 2: Each point is on at least two lines.

Each point is on an infinite number of lines.

Theorem 3: There is a triple of lines that do not share a common point.

FE, GC, and AD for example.

Lot's of Theorems!

Ok now for the other two we'll be looking at all semester:

Spherical and Hyperbolic

Spherical Geometry is the geometry on the surface of a ball (notably the Earth). Hyperbolic Geometry is the geometry of outer space as far as what we know so far.

They each have an axiomatic system with all the parts to work with. Point, line, plane, on…these are some of the undefined terms for each.

Now Hyperbolic Geometry has 2D models and 3D models just like Euclidean G. Spherical only has 3 D models.

Hyperbolic Geometry

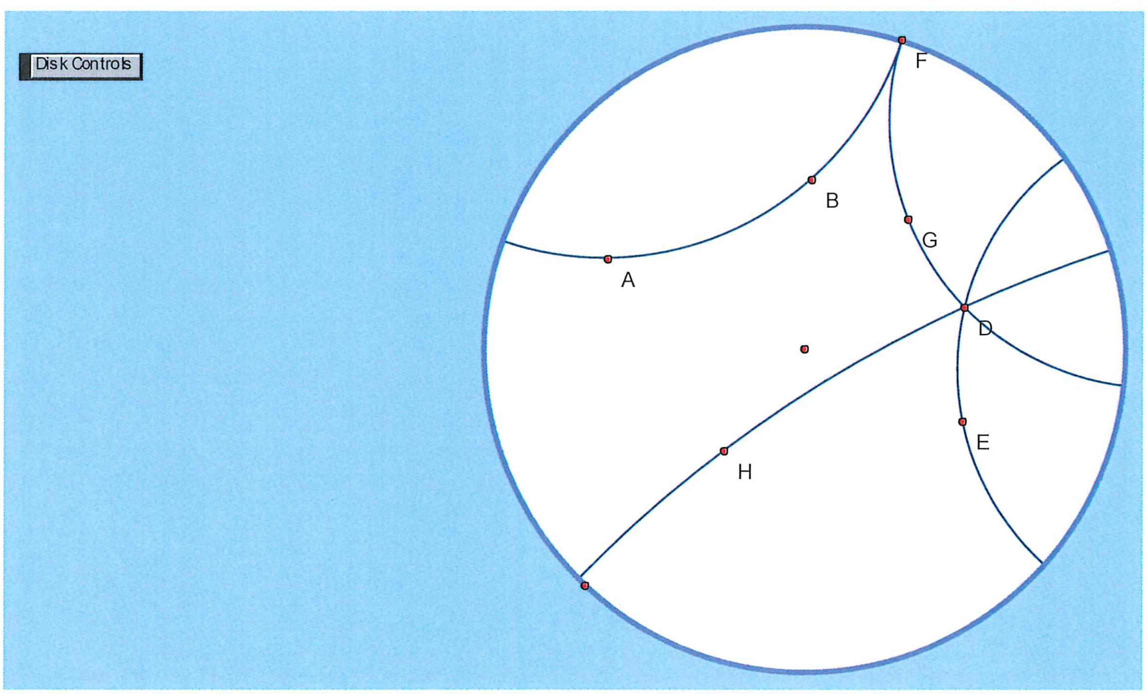
We will look at a (a not the!) 2D model for Hyperbolic G. It's the unit circle just like in the Klein Disc BUT it has a distance formula that makes the distance inside the disc infinite. Lines are special curves called Orthogononal circles.

Parallel Lines in Hyperbolic Geometry, Poincare disc model. The BOLD circle is not in our space! It's the interior points only. so F O is not in our space

SMSG A16 The Parallel Postulate:

> Through a given external point there is at most one line parallel to a given line.

Pick line AB and Point D.

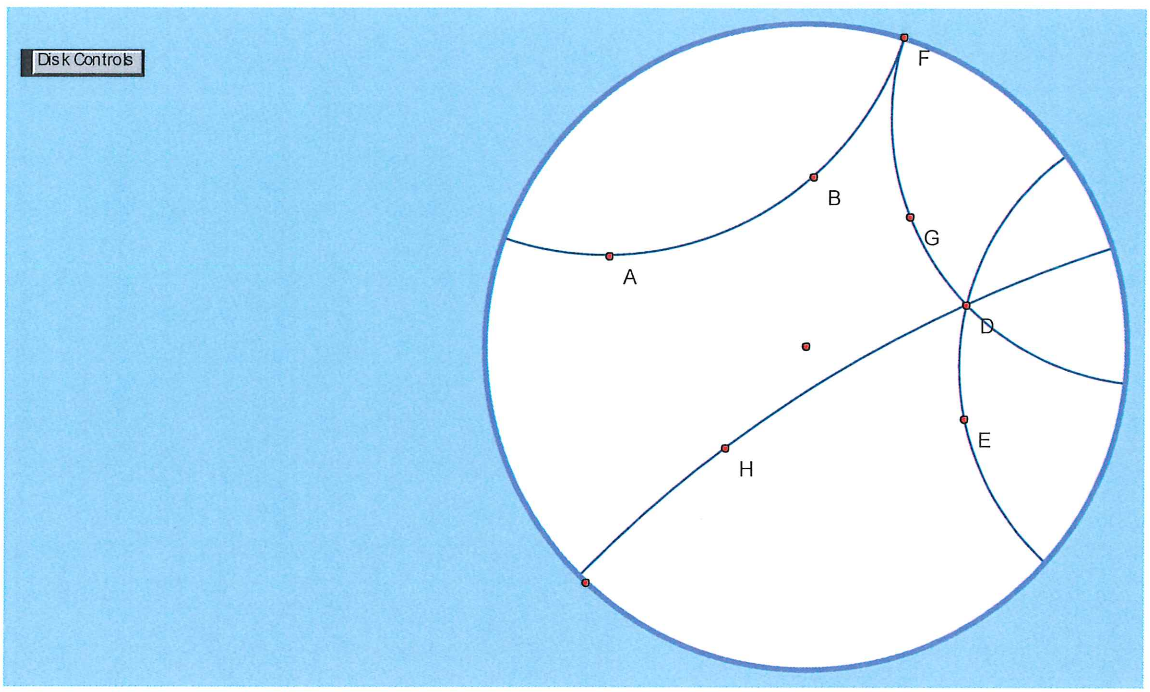


H $\overline{AB}$ is parallel to every other line showing in the disc.

Since H $\overline{AB}$ intersects H $\overline{DF}$ on the circle, these two have a type of parallelism called "asymptotically parallel".

H$\overline{DH}$ and H$\overline{DE}$ are "divergently parallel" to H$\overline{AB}$.

So we have H$\overline{AB}$ and a point not on it: Point D and we have 3 lines parallel to H $\overline{AB}$ through D right there on the sketch. This illustrates our choice of parallel axiom. And we now have two types of parallelism: asymptotic and divergent, again with this geometry.

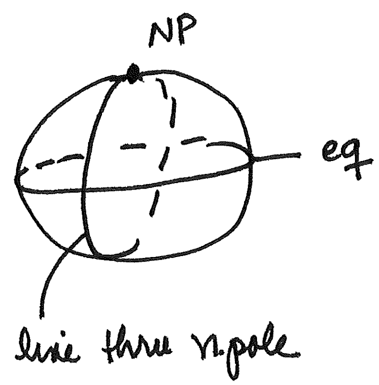


I'll show you many pictures as we move along!

Now let's look at Spherical Geometry

Spherical G is modeled by the unit sphere and the surface points of the sphere are points in the space. Lines are Great Circles like the equator, but not limited to the equator. Lines through the north and south poles are Great Circles. Grab a ball and 3 rubber bands. Put on an "equator", then run a band through the north and south pole. Then run the third band through the poles offset from the first one. Note that a plane through a great circle INCLUDES the center of the circle. Other lines of latitude make circles and not lines.

No software available. I am NOT an artist!



Do you see that lines (Great Circles intersect in TWO points? And that more than one line can go through two points? Very Non-Euclidean.

Pick the equator and the North Pole.

SMSG A16 The Parallel Postulate:

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>
> How many rubber bands go through the North pole and DON't intersect the equator?

More to come on these two!

Popper 1.1, Question 11

There are geometries with no parallel lines with our "pick a line and a point not on that line" setup.

A. True

B. False

Popper 1.1, Question 12

Given a line and a point not on that line, some geometries have more than one line through the point sharing no points in the space with the given line.

A. True

B. False

Almost done

Essay 1.1, Number Two

What do you think about the fact that multiple geometries exist? Spend some time on the internet researching the history of geometry. Write a 3 paragraph one page, front side only, typed paper (14 point type) discussing how you feel about what you found out. Be sure to put in quotation marks and cite info on anything that you need to.

Ok for work

Popper 1.1 11 questions

2 essays

EMCF

assignments
1 page front, typed

no homework yet